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THE MECHANICS OF BENT YARNS

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FOREWORD

This report is one of a series of studies sponsored by the Quartermaster Research & Development Laboratories in the general field of the mechanical properties of textile structures. In previous reports of this Textile Series (Nos. 60 and 62) attention has been devoted to the effects of form factors on the translation of the inherent physical properties of textile fibers into yarn structures. Both staple and continuous filament yarns have thus far been studied in a straight configuration. It is therefore logical to devote attention to the geometry of bent yarns, for it is in this form that yarns lie in the end textile products.

It is the purpose of this report to formalize the elements of bent yarn geometry through careful analysis of an idealized yarn structure. The products of such an analysis are expressed in mathematical relationships presented in a form most useful to the textile materials engineer. To encourage use of the results by practical designers, an effort has been made to present complete graphical conversions from practical textile data to the dimensionless parameters utilized for maximum generality in the analysis.

This study on bent yarn geometry, conducted by Mr. Stanley Backer, consultant to the Quartermaster Laboratories, forms a logical part of the broad continuing textile program sponsored by the Quartermaster Corps in the field of stress-analysis in textile structures.

The work was done as part of one of the projects of the Textile Materials Engineering Laboratory in Philadelphia, which is headed by Mr. Louis I. Weiner. The assistance of Mr. Walter J. Hynek, Physicist of this laboratory, in computing values for many of the equations and in the preparation of the graphs is gratefully acknowledged.

S. J. KENNEDY
Research Director
for
Textiles, Clothing and Footwear

May 1952

Introduction

The influence of yarn geometry on the translation of fiber properties has received increasing attention during recent years. The early studies reported emphasized the relationship between the structure and the mechanical properties of singles yarns whose axes lie on a straight line. [4,5,7,8] Of late the importance of bending in yarn geometry and in the mechanical properties of the end textile structure has been clearly indicated. [1,6] For the yarns we use are rarely straight; they are usually bent into torus shapes during the weaving process or into helical forms when twisted into plied structures. Yarn bending is inherent in woven and knitted structures, in sewing threads, tire cords, twines, and ropes. In addition yarns are subjected to bending during use of the end textile structure, in flexing, draping, or creasing of an apparel fabric, in the waving of a flag, the billowing of a tent, the packing of a sleeping bag, the knotting of a rope, the forming of a stitch in the sewing process, the flexing of a rope over a pulley, the washing of a blanket.

The mechanical properties of the original textile structure and its behavior under service conditions will depend to a great extent upon the strain to which its individual fibers are subject. The level of fiber strain will in turn depend on the configuration which the fiber is forced to assume, and therefore on the structure of the bent yarn. It follows that study of bent yarn geometry is essential to the full understanding of strain and therefore of stress distributions in the textile structure. Only through such understanding can we hope to predict the integrated mechanical behavior of twisted, woven, or knitted textile products.

This report attempts to outline the development of an idealized geometry of bent yarns. It is realized that actual yarn structure will at times depart to a significant degree from the mathematical forms proposed herein. However, even in such cases the geometric forms will be useful as a common basis of structural comparisons, and considerable insight into the behavior mechanisms in bending of a given material will follow from study of the causes of its deviations from the ideal geometry. The implications of such insight has direct bearing on long-standing problems in many areas of textile research. We refer to such subjects as knot efficiencies, seam efficiencies, sewability (from the standpoint of loop formation), cord and power transmission, rope wear, flex resistance and internal abrasion of fabrics, crease and muss resistance and abrasion resistance.

It should not be supposed that the following material is solely the work of the author. A major portion of the analysis here proposed is based on work of others, notably Schwarz [1] and Chow. [2] The modus operandi of this study involves 1) selection of special cases from the more general studies of twist in plied structures and 2) the extension of these special cases as they relate to the geometry of bent singles yarns. Specifically the report deals with: A. the local strain caused in fibers at various yarn locations when they are bent, B. the average strain taking place in fibers forced to follow the idealized geometric path in bending without the benefit of freedom of relative motion with the yarn, C. the relative motion between fibers due to shifting from the bottom to the top part of the loop, D. the relative motion between fiber surfaces due solely to changes in local helix angles, E. the changes in local helix angles which occur in yarn bending, and F. the curvatures of fibers at all locations in a bent yarn.

To achieve maximum generality in the analytical development it has been found expedient to use dimensionless parameters wherever possible. This feature will, however, reduce the speed of computation when basic data are available in common textile terms rather than in the geometric parameters used throughout the report. This difficulty can be alleviated by provision of graphical conversions for use of the practical designer. Further ease in computation is made possible through provision of the graphical form of the analytical relationships, plotted over practical ranges of the geometric parameters. The analysis follows.

A. Local Strain

A singles yarn is taken as the major element of structure. The yarn is made of continuous filaments which are considered to lie on right cylindrical surfaces along circular-helical paths. The cylindrical surfaces are concentric and spaced one fiber diameter apart. All fibers on all surfaces are subject to the same number of turns about the yarn axis per unit length along the yarn axis. It follows that helix angles formed by the fibers at each concentric surface will vary, with the maximum helix angle occurring at the outermost fiber and zero helix angle at the yarn axis. It is assumed that differences in path length of filaments at different concentric surfaces are accounted for in the spinning or twisting operation without the necessity of fiber tension or compression. The local helix angle for a given surface is constant for all fibers on that surface and

$$\tan Q_v = 2\pi vt \qquad (1)$$

where Q_v is the fiber helix angle on the cylindrical surface of radius v and T is the twist per inch of the yarn. In particular, the outer fibers of the yarn form the helix angle, Q, with the yarn axis and if d is the yarn diameter

$$\tan Q = \pi dT \qquad (2)$$

The single yarn described above (taken to be a warp yarn for convenience in nomenclature) is assumed to form a torus when bent around a filling yarn, as illustrated in Figure 1. The radius of the bent warp yarn is designated as a and the radius of the torus as r. The torus radius, r, equals the sum of the radii of the warp (crown) yarn and the filling (inner) yarn.

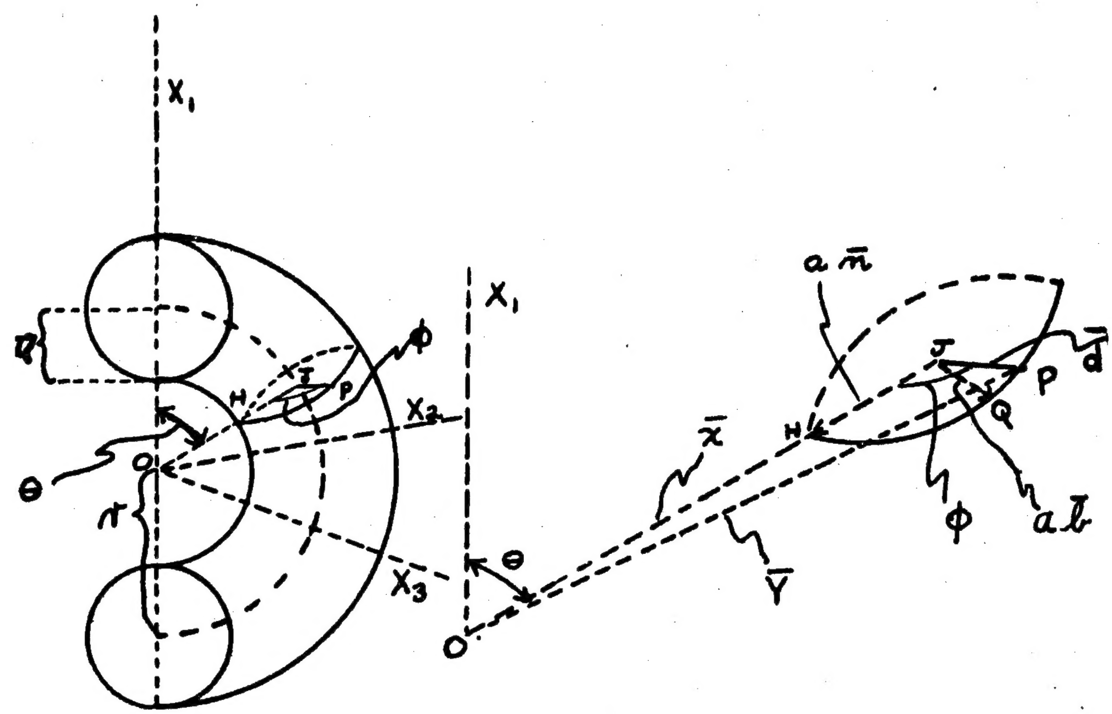


Fig. 1. Torus Form of Bent Yarns.

Let O be the torus center and let P be a point on a fiber as it twists around the warp yarn. Let X_1, X_2, and X_3 be mutually orthogonal axes forming the cartesian coordinate system. Let X_3 be the axis of the torus, i.e. the axis of the filling cross yarn. The circular path of the warp yarn lies in the X_3 plane. The intersection of a plane through X_3, making an angle θ with X_1, is indicated in Figure 1 as passing through point P. The circular intersection of the torus and indicated plane is enlarged on the right side of Figure 1. Let J be the yarn axis at the circular section. Let OJ intersect the circle at H. Draw JQ in the plane of the section, perpendicular to JH.

Designate the vector OJ as $\bar{X}$, OP as $\bar{Y}$, HJ as a $\bar{n}$, JQ as a $\bar{b}$ and JP as $\bar{d}$. The vector $\bar{n}$ is a unit vector coinciding with, but in opposite direction to vector $\bar{X}$. Vector $\bar{n}$ is the principal normal to the space path of the warp yarn axis. The vector $\bar{b}$ is also a unit vector, is perpendicular to n, and is called the binormal of the warp axis space path. Angle θ lies between the X_1 coordinate and the vector $\bar{X}$. Angle ϕ lies between vector $\bar{X}$ and vector $\bar{d}$. The ratio between θ and ϕ is assumed to be constant and is designated as λ, i.e.

$$\phi = \lambda \theta \tag{3}$$

This simply means that in proceeding along the warp yarn axis (in the torus configuration) through an angle θ about the torus axis, the fiber twists around its yarn axis through a corresponding angle ϕ. For each revolution around the torus axis $(\theta = 2\pi)$ there are λ revolutions or turns of the fiber about the warp yarn axis.

Consider the components of the above named vectors:

$$\begin{array}{ll} \bar{X} & [r\cos\theta, \; r\sin\theta] \\ \bar{b} & [0, \; 0, \; 1] \\ \bar{n} & [-\cos\theta, \; -\sin\theta, \; 0] \end{array} \tag{4}$$

Now $\bar{d} = a(\cos\phi)\,\bar{n} + a(\sin\phi)\,\bar{b}$, $\quad$ (5)

therefore $\bar{Y} = \bar{X} + \bar{d} = \bar{X} + a\cos\phi\,\bar{n} + a\sin\phi\,\bar{b}$ $\quad$ (6)

or $\bar{Y} \left[Y_1, Y_2, Y_3\right]$ can be written as

$$Y_1 = r \cos\ \theta + a \cos\ \phi\ (-\cos\ \theta\ )$$

$$Y_2 = r \sin\ \theta + a \cos\ \phi\ (-\sin\ \theta\ )$$

$$Y_3 = a \sin\ \phi$$

and substituting (3)

$$Y_1 = r \cos\ \theta - a \cos \lambda\theta\ \cos\ \theta$$

$$Y_2 = r \sin\ \theta - a \cos \lambda\theta\ \sin\ \theta$$

$$Y_3 = a \sin \lambda\theta \qquad (7)$$

If dS is taken as a differential element of arc length along the fiber passing through point P it follows from differential geometry that

$$\frac{dS}{d\theta} = \sqrt{\sum \frac{dY_i}{d\theta} \quad \frac{dY_i}{d\theta}} \qquad (8)$$

Following Chow's [2] analysis an expression is now derived for the element of arc length, dS, in terms of the torus geometry. Note this is but a special case of Schwarz's consideration [1] of the twist structure of plied yarns. For first derivatives:

$$\frac{dY_1}{d\theta} = -r \sin\ \theta + a \sin\ \theta\ \cos \lambda\theta + a\ \lambda \sin\lambda\theta\ \cos\theta$$

$$\frac{dY_2}{d\theta} = r \cos\ \theta - a \cos \lambda\theta\ \cos\ \theta + a\lambda \sin \lambda\theta\ \sin\ \theta \qquad (9)$$

$$\frac{dY_3}{d\theta} = a\lambda \cos \lambda\theta$$

Squaring each equation in (9) and adding, we have from (8) the differential arc length

$$dS = \sqrt{(a \cos \lambda \theta - r)^2 + a^2 \lambda^2} \; d\theta \qquad (10)$$

or

$$dS = \sqrt{\frac{(r - a \cos \phi)^2}{\lambda} + a^2} \; d\phi$$

By integration of dS through a complete turn of the fiber about the yarn axis it has been shown that [1], [2] the length of the fiber loop in the bent yarn equals* that of the fiber loop in the yarn before bending. It follows that no tensile strain will take place in the individual fibers of a yarn upon bending if they are allowed to redistribute their lengths within individual loops about the yarn axis. This assumption was made by Czitary in his analysis of bending strains of wire cables [9]. However, if excessive friction prevents the slightest redistribution of length within the loop, differences in local fiber path lengths must result in local tensile and compressive fiber strains.

Axial fiber strain, $\mathcal{E}$, is defined, in the case of yarn bending, as

$$\mathcal{E} = \frac{dS - dS_o}{dS_o} \qquad (11)$$

where dS is the fiber path length in the angular increment $d\phi$ when the yarn is bent and dS_o is the corresponding arc length in the same $d\phi$ for the straight yarn. It is here assumed that during the bending process no new fiber material is introduced into the angular increment $d\phi$ because of the restriction on fiber movement. The arc length dS_o is determined from

$$\frac{a d\phi}{dS_o} = \sin Q \qquad (12)$$

and λ from

$$\lambda = 2\pi r T = 2\pi r \left(\frac{\tan Q}{2\pi a}\right) = \frac{r \tan Q}{a} \qquad (13)$$

---

*Within well-defined limits.

Substituting (12) and (13) in (10) and (11)

$$\varepsilon = \sqrt{\frac{\dfrac{(r-a\cos\phi)^2}{\dfrac{r^2\tan^2 Q}{a^2}} + a^2 \quad d\phi}{\dfrac{a\,d\phi}{\sin Q}}} - 1$$

$$\varepsilon = \sqrt{\frac{\left(\frac{r}{a} - \cos\phi\right)^2 + \lambda^2}{\left(\frac{r}{a}\right)^2 + \lambda^2}} - 1 \qquad (14)$$

Values of ε are plotted in Figure 2 against ϕ as ϕ varies from 0 to 180°. Symmetry conditions apply in the range ϕ equals 180° to 360°. Parameter r/a varies from 1.5 to 2.5, the normal range of diameter variation encountered in practical fabrics. Parameter λ is varied from .5 to 1.5. If F is defined as the twist multiplier,

$$T = F\sqrt{N} \qquad (15)$$

where T is the twist in turns per inch, and N is the yarn count. If the specific volume of the yarn is taken to be 1.1 so that, following Peirce [4],

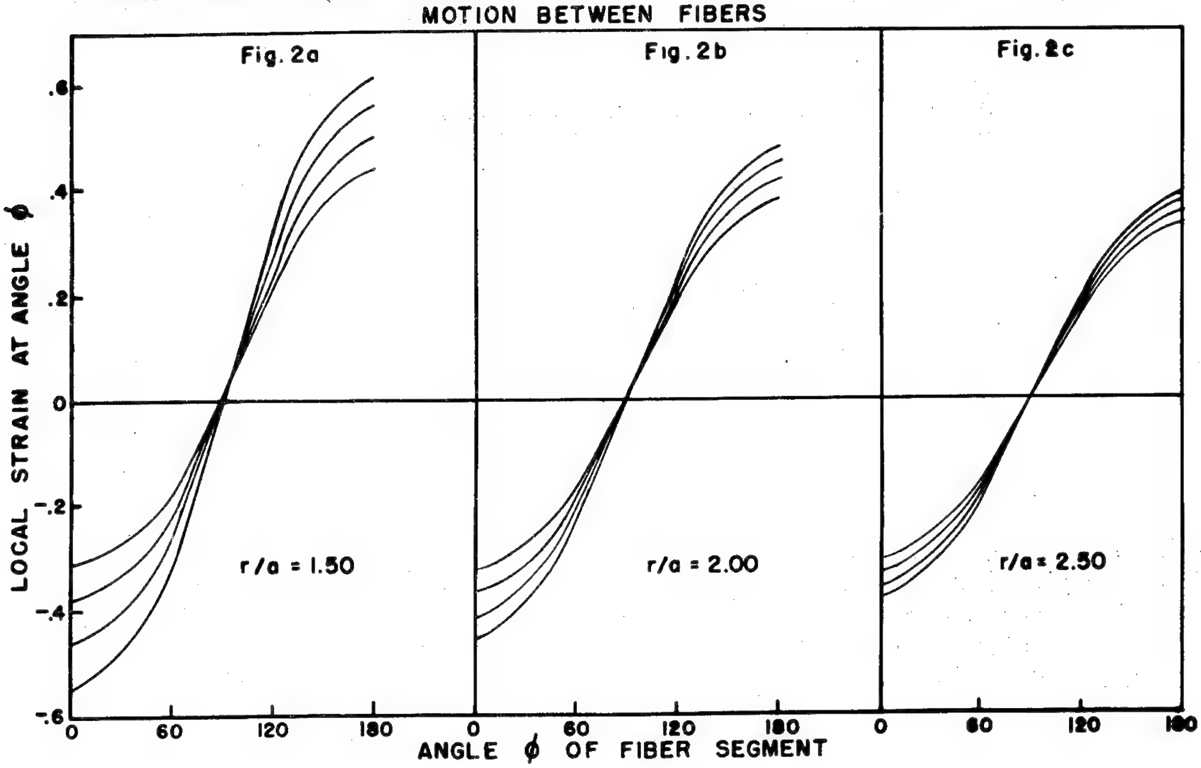
$$d = 2a = \frac{1}{K\sqrt{N}} \qquad (16)$$

where K depends on the yarn system used (K = 28 for cotton, 15.4 for woolen cut, 22.8 for worsted), then

$$\lambda = \left(\frac{r}{a}\right)\frac{\pi F}{K} \qquad (17)$$

FIG. 2

LOCAL FIBER STRAIN CAUSED BY YARN BENDING ASSUMING NO FREEDOM OF MOTION BETWEEN FIBERS



For r/a equal to 2, that is where warp and filling yarns are of equal diameter, the range 'λ equals .5 to 1.5' represents a range in twist multiplier from 'F equals 2.23 to 6.70' in the cotton system, '1.23 to 3.69' for woolen cut and '1.38 to 5.45' in the worsted system. It is evident in Figure 2 that ϵ is 0 at $\phi = \pi/2$ and reaches a maximum in tension (positive strain) at $\phi = 180^\circ$ and a maximum in compression (negative strain) at $\phi = 0^\circ$. As the relative twist, represented by λ, is increased, ϵ is decreased. As the ratio r/a increases ϵ decreases.

B. Average Strain

In an actual yarn it is probable that local slippage and length redistribution will take place before strains of the magnitude plotted in Figure 2 are reached. A parameter of a more practical nature is the average strain which takes place in tension or that which takes place in compression. It is evident from Figure 2 that the fiber path between $\phi = \pi/2$ and π is in tension in the bent yarn, while between $\phi = 0$ and $\pi/2$ is in compression. It remains to determine the path length along the outer segment of the path, and along the inner segment, and to compare these lengths with the helical fiber path of the unbent yarn. As has been indicated (1) (2) the length of the fiber loop in the bent yarn equals that of the fiber loop in the yarn before bending. It follows that the overall extension of the outer path must equal the overall extension of the inner path, since both have the same length in the unbent yarn, i.e. equal half a helical loop. Now let ΔS be the difference in path length in the outer and inner parts of the bent yarn fiber loop. By symmetry we can deal with

$$\frac{\Delta S}{2} = \int_{\phi = \frac{\pi}{2}}^{\phi = \pi} dS - \int_{\phi = 0}^{\phi = \pi/2} dS \qquad (18)$$

the incremental distance can be expressed as

$$dS = \sqrt{(r - a \cos \lambda\theta)^2 + a^2 \lambda^2}\; d\theta = a\lambda \sqrt{\left[\left(\frac{r}{a\lambda}\right) - \frac{\cos \lambda\theta}{\lambda}\right]^2 + 1}\; d\theta \qquad (19)$$

Upon integration of (18)

$$\frac{\Delta S}{2} = \frac{2\,ar}{\lambda\sqrt{a^2\lambda^2 + r^2}} - \frac{2a^5 r\lambda}{3\left(\sqrt{a^2\lambda^2 + r^2}\right)^5} \qquad (20)$$

Now the average tensile strain in the top loop

$$\overline{\varepsilon_T} = \int_0^{\frac{\Delta S/2}{\pi}} dS \tag{21}$$

From the geometry of the straight yarn

$$\int_0^{\pi} dS = \frac{\pi\sqrt{a^2\lambda^2 + r^2}}{\lambda} \tag{22}$$

which agrees with Chow's derivation [2] of the length of λ loops

$$S = 2\pi\left(a^2\lambda^2 + r^2\right)^{1/2} \tag{23}$$

therefore

$$\overline{\varepsilon_T} = \frac{\dfrac{2\,ar}{\lambda\sqrt{a^2\lambda^2 + r^2}} - \dfrac{2\,a^5 r\lambda}{3\left(\sqrt{a^2\lambda^2 + r^2}\right)^5}}{\dfrac{\pi}{\lambda}\left(a^2\lambda^2 + r^2\right)^{1/2}} \tag{24}$$

$$\overline{\varepsilon_T} = \frac{2\,ar}{\pi\left(a^2\lambda^2 + r^2\right)} - \frac{2\,a^5 r\lambda^2}{3\pi\left(a^2\lambda^2 + r^2\right)^3} \tag{25}$$

In Figure 3 the relationship between $\Delta S/2$ and λ is presented graphically for various r/a ratios. For the purpose of this computation $\Delta S/2$ has been set equal to the first term in (20), or

$$\frac{\Delta S}{2} \cong \frac{2r}{\lambda\sqrt{\lambda^2 + (r/a)^2}} \tag{26}$$

It is seen in Figure 3 that ΔS decreases with increases in λ or (r/a). Omission of the second term of (28) causes less than 1% error for the case where $r/a = 2$ and $\lambda = 1$. In Figure 4, $\bar{\xi}$ is plotted against λ for various r/a ratios. For the purpose of this computation $\bar{\xi}$ is set equal to the first term of (33),

or

$$\bar{\xi} \cong \frac{2\ r/a}{\pi\left[\lambda^2 + \left(\frac{r}{a}\right)^2\right]} \tag{27}$$

Again an error of less than 1% is introduced by this approximation for the case $r/a = 2$ and $\lambda = 1$.

The practical utility of the quantities $\bar{\xi}$ and $\Delta S/2$ must of necessity be limited by the degree to which actual yarns follow the geometric assumptions of the derivation. In a general sense, however, $\bar{\xi}$ gives a comparative value which may find use in the design of more durable twisted structures in which fiber or strand movement is restricted. In structures which allow complete freedom of movement, the quantity <u>$\Delta S/4$ represents the length of fiber which passes the point $\phi = \pi/2$ in the redistribution of material from lower to upper parts of the helix loop</u>.

C. Local Helix Angle

It has been indicated that the helix angle Q_v of a straight singles yarn is dependent (1) (2) on the local radius, v, of the yarn and on its twist. This angle Q_v is therefore constant for a given v and T at every point along the yarn. However, when the yarn assumes the bent form of Figure 1 the local helix angle is no longer a function of v and T alone, as has been indicated by

Fig. 3

DIFFERENCE IN PATH LENGTH OF UPPER & LOWER PARTS OF THE FIBER LOOP INDICATED AS A FUNCTION OF r/a AND λ. HERE r IS TAKEN AS THE UNIT OF LENGTH & ΔS/2 IS EXPRESSED IN TERMS OF THIS UNIT

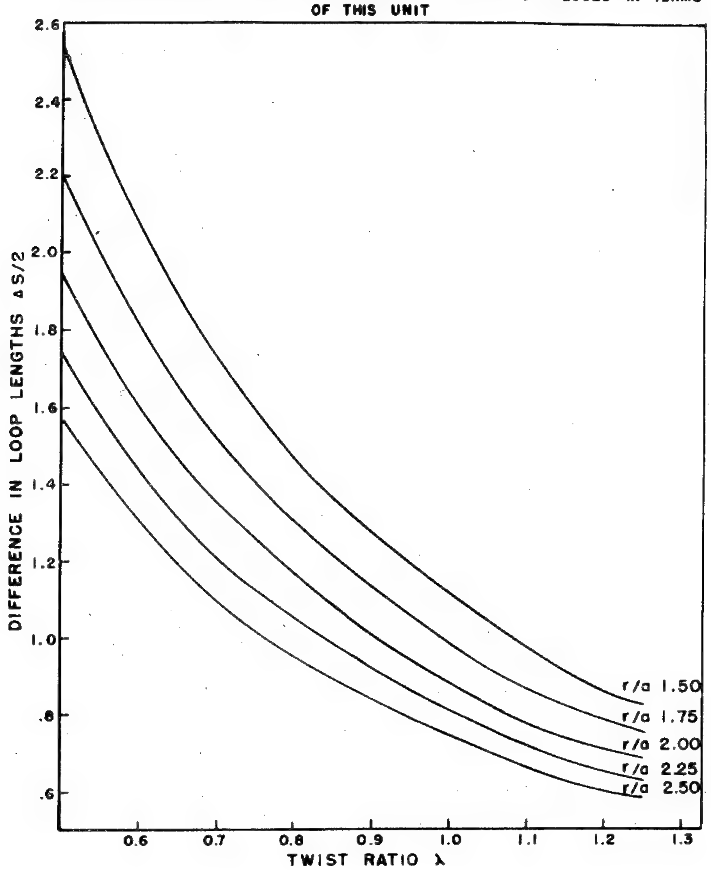
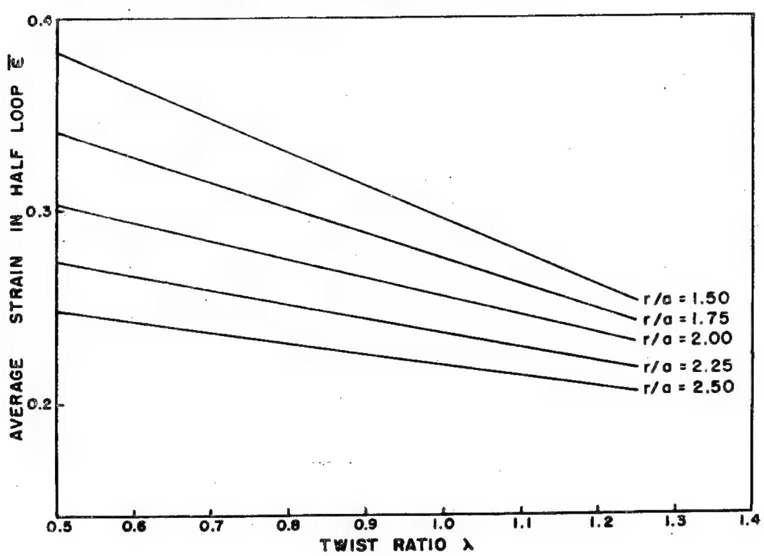


FIG. 4

AVERAGE STRAIN IN HALF LOOP DUE TO DIFFERENCE IN PATH LENGTH FOR VARYING λ AND r/a.



Chow [2] for the general case of the singles yarn as it lies in the ply. Following the method used for the ply yarn (2) we derive the relationship between the local helix angle and the parameters λ, r/a and ϕ for the special case of the bent singles.

In Figure 1, $\bar{X}$ represents the vector connecting the origin and a point on the path of the yarn axis, while $\bar{Y}$ is the fiber path vector. To determine the local helix angle we first derive expressions for $\bar{t}$, the unit tangent vector to the path of the yarn axis at the section (point J) indicated in Figure 1; and $\bar{T}$, the unit tangent vector to the path of the fiber at the same section, i.e. point P. It follows from

$$\bar{t} \cdot \bar{T} = |\bar{t}||\bar{T}| \cos\alpha \qquad (28)$$

that

$$\cos\alpha = \bar{t} \cdot \bar{T} \qquad (29)$$

where α is the local helix angle.

From Frenet $\dfrac{d\bar{X}}{ds} = \bar{t}$; $\qquad \dfrac{d\bar{Y}}{dS} = \bar{T} \qquad (30)$

where s is the distance along the yarn axis and S is the distance along the fiber as it lies in the bent yarn.

First for $\bar{t}$

$$\bar{t} = \left[t_1, t_2, t_3\right] = \frac{d\bar{X}}{d0} = \frac{d\bar{X}}{rd\theta}$$

$$= \left[\frac{d(r\cos\theta)}{r\,d\theta}, \frac{d(r\sin\theta)}{r\,d\theta}, 0\right] \qquad (31a)$$

$$= \left[-\sin\theta, \cos\theta, 0\right]$$

From (30) (10) and (8)

$$\bar{T} = \frac{d\bar{Y}}{dS} = \frac{d\bar{Y}}{d\theta}\frac{d\theta}{dS} = \frac{d\bar{Y}}{d\theta}\frac{1}{\sqrt{\sum \frac{dY_i}{d\theta}\frac{dY_i}{d\theta}}} \qquad (31b)$$

Now,

$$\bar{t} \cdot \bar{T} = \frac{(r-a\cos\lambda\theta)}{\sqrt{(r-a\cos\lambda\theta)^2 + a^2\lambda^2}} = \frac{1}{\sqrt{1 + \dfrac{a^2\lambda^2}{(r-a\cos\lambda\theta)^2}}} = \cos\alpha \qquad (32)$$

or

$$\tan\alpha = \frac{a\lambda}{(r-a\cos\lambda\theta)} = \frac{a\lambda}{(r-a\cos\phi)} \qquad (33)$$

In Figure 5 a graphical relationship between α and λ is presented for the inner, outer, and center points of the bent yarn ($\phi = 0, \pi$, and $\pi/2$) and for various r/a ratios. Clearly α is larger at $\phi = 0$ than at any other angle, and is a minimum at $\phi = \pi$. At $\phi = \pi/2$

$$\tan\alpha = \frac{a\lambda}{r} = \frac{a\,2\pi r T}{r} = 2\pi aT \qquad (34)$$

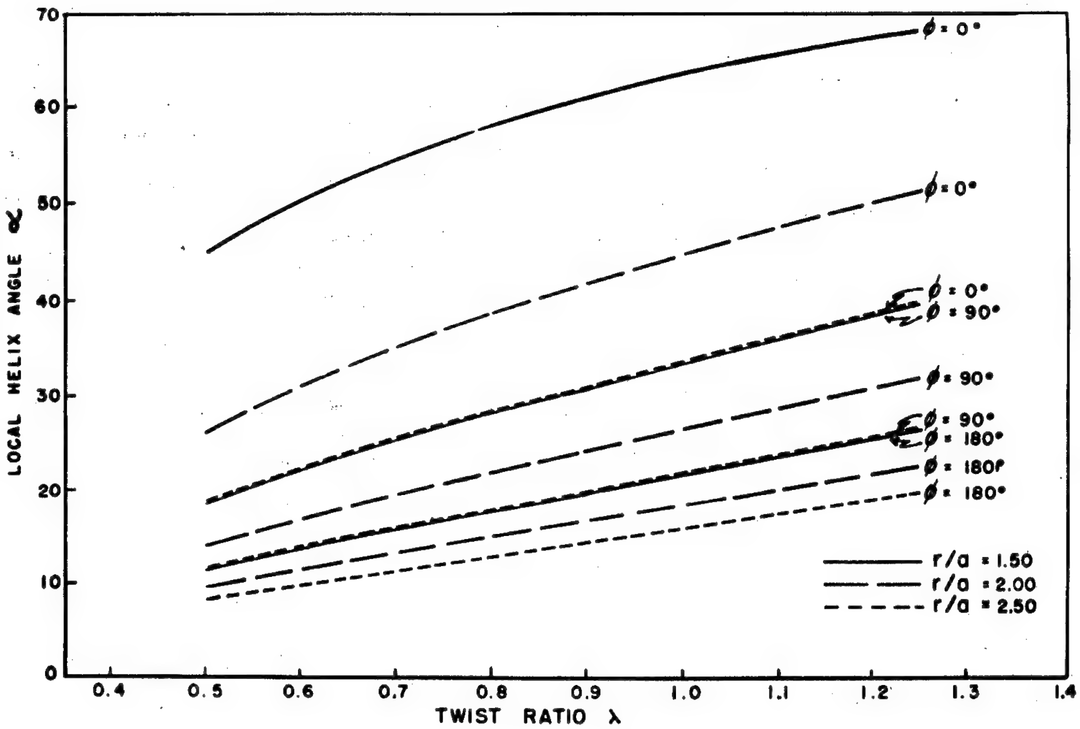
which is the value of tan Q in the unbent yarn. No helix angle change will take place along the 'neutral axis' of a yarn during bending while maximum changes will be effected at $\phi = 0$ and π.

A change in the local helix angle will cause some relative movement between the surfaces of adjacent fibers -- a factor of importance in studies of the internal wear and friction or felting of yarns and fabrics. The extent of this movement may be demonstrated easily with the aid of Figures 6 and 7. Figure 6 demonstrates the appearance of fibers in the singles yarn before bending and after bending, when viewed from the bottom and top of the yarn crown. The dark horizontal line drawn across the fibers in Figure 6a is rotated counterclockwise in segments in Figure 6b and clockwise in 6c. These segments are indicated as aa' in Figure 7, while bb represents the fiber diameter, and $\overline{Ob}$ or ρ is the fiber radius. The relative movement between two points originally in contact on adjacent fibers will therefore be $2\,\overline{ae}$ where

$$2\,\overline{ae} = 2\rho\,(\cot\alpha - \cot\alpha_o) \qquad (35)$$

$$2\,\overline{ae} = \frac{-2\rho\cos\phi}{\lambda}$$

Fig. 5
LOCAL HELIX ANGLES IN BENT YARNS



To effectively demonstrate the change of helix angle at the inner and outer points of a yarn bend, Figures 8 and 9 have been included [3], [7]. These illustrate the more general singles yarn bend as it exists in the ply of a wire rope and of a three ply nylon yarn. Here ellipticity of the filament or strand section reflects the relative helix angles. Practical considerations of the local helix angle will be taken up in a later section.

D. Local Curvature

For the case where complete freedom of fiber or strand movement exists, local strains occurring in the bending process will depend solely on the local fiber or strand curvature. Computation of these local strains has been reported by several workers in the field of civil engineering, [9] but their work has been confined

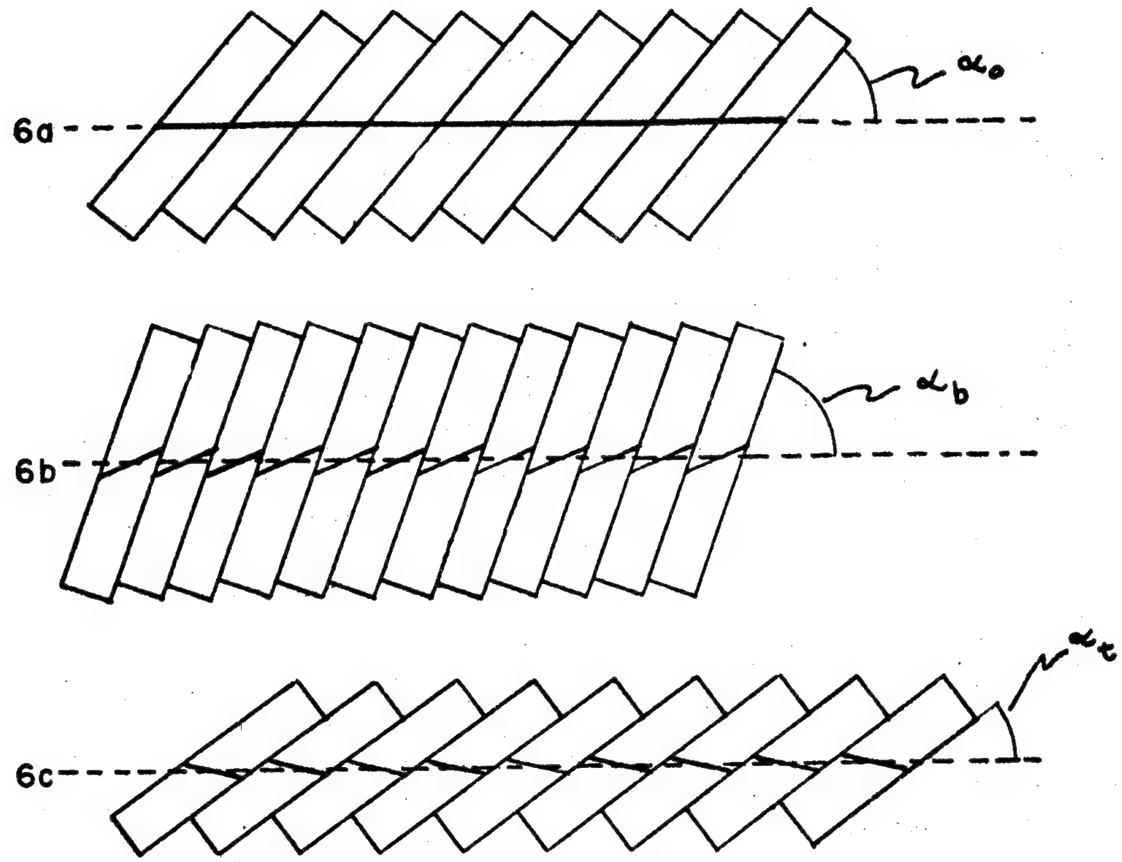


Fig. 6. Change in Local Helix Angles Due to Yarn Bending.

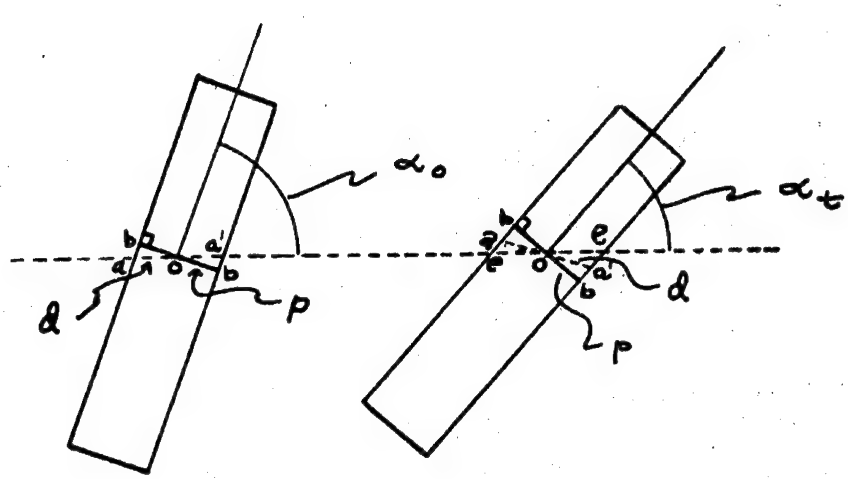


Fig. 7. Relative Movement Between Fibers Due to Yarn Bending.

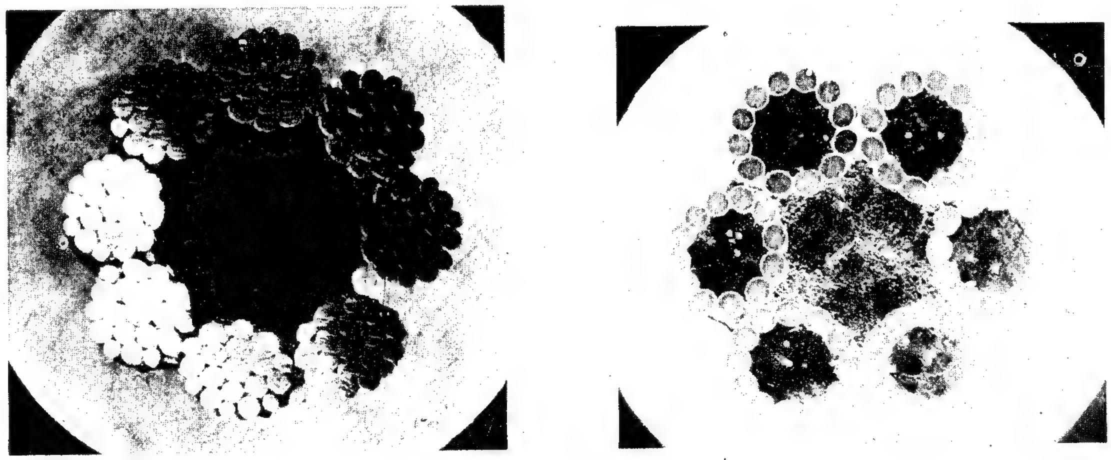


Fig. 8. Cross Sections of Wire Ropes Corresponding to Plied Yarns. (Courtesy of E.R. Schwarz)

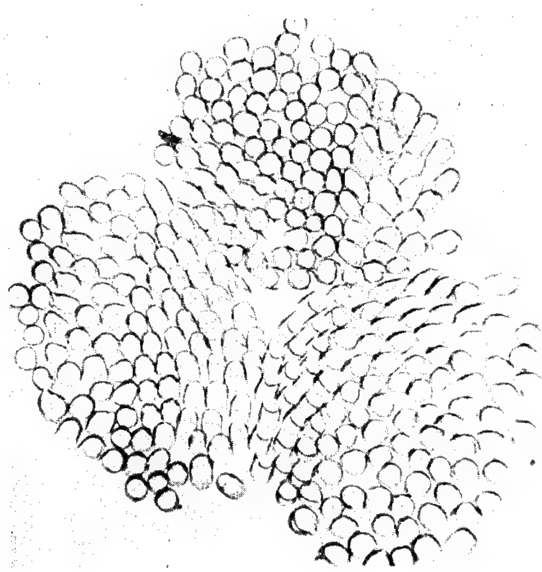


Fig. 9. Cross section of 3 ply 102 filament nylon yarns. Note fibers at center of ply lie at a steep angle with the ply axis as evidenced by their elliptical appearance (Courtesy of E. R. Schwarz.)

for the most part to low helix angles typical of wire cable construction. Here we deal with much higher helix angles than are employed in the previous work and must therefore derive more exact expressions for local curvatures.

From Frenet

$$\frac{d\bar{T}}{dS} = k\bar{n} \tag{36}$$

where $\bar{T}$ is the unit tangent to the fiber at point P in Figure 1, dS is the increment of fiber length, $\bar{n}$ is the principal normal to the fiber path at P and k is the local curvature of the fiber at P. Let

$$u = (a \cos \lambda \theta - r)^2 + a^2 \lambda^2 \tag{37}$$

then from (19) and (31b)

$$\bar{T} = \frac{d\bar{Y}}{dS} = \frac{d\bar{Y}}{d\theta}\frac{d\theta}{dS} = \frac{d\bar{Y}}{d\theta}\frac{1}{\sqrt{u}} \tag{38}$$

Now

$$\frac{d\sqrt{u}}{d\theta} = \frac{1}{2} u^{-\frac{1}{2}} \frac{du}{d\theta} = \frac{1}{2} u^{-\frac{1}{2}} 2(a \cos \lambda \theta - r)(-a \lambda \sin \lambda \theta)$$

$$= \frac{-(a \cos \lambda \theta - r) \; a \lambda \sin \lambda \theta}{\sqrt{(a \cos \lambda \theta - r)^2 + a^2 \lambda^2}} \tag{39}$$

and

$$k\bar{n} = \frac{d\bar{T}}{d\theta}\frac{d\theta}{dS} = \frac{d}{d\theta}\left[\frac{d\bar{Y}}{d\theta}\frac{1}{\sqrt{u}}\right]\frac{d\theta}{dS} \tag{40}$$

$$\bar{kn} = \frac{\sqrt{u}\;\frac{d^2\bar{Y}}{d\theta^2} - \frac{d\bar{Y}}{d\theta}\,\frac{d\;\sqrt{u}}{d\theta}}{u}\;\frac{1}{\sqrt{u}}$$

$$= \frac{u\;\frac{d^2\bar{Y}}{d\theta^2} + \frac{d\bar{Y}}{d\theta}\,(a\cos\lambda\theta - r)\,(a\lambda\sin\lambda\theta)}{u^2} \tag{41}$$

Taking the dot product of $\bar{Kn}\;\cdot\;\bar{Kn}$ we obtain (taking $g = r/a$)

$$k = \frac{\sqrt{(\cos\lambda\theta - g)^2 + \lambda^4 + 2\lambda^2\cos\theta(\cos\lambda\theta - g) + 4\lambda^2\sin^2\theta - \frac{\lambda^2\sin^2\lambda\theta(\cos\theta - g)^2}{(\cos\lambda\theta - g)^2 + \lambda^2}}}{a\left[(\cos\lambda\theta - g)^2 + \lambda^2\right]} \tag{42}$$

This expression is greatly simplified for the case where $\phi = \theta$ and $\cos\phi = \cos\lambda\theta = \pm 1$ and $\sin\lambda\theta = 0$ so that

$$k = \frac{\sqrt{(1-g)^2 + \lambda^4 + 2\lambda^2(1-g)}}{a\left[(1-g)^2 + \lambda^2\right]} = \frac{\left[(1-g) + \lambda^2\right]}{a\left[(1-g)^2 + \lambda^2\right]} \tag{43}$$

At $\phi = \pi$, $\cos \phi = -1$; $\sin \lambda \theta = 0$

$$k = \frac{\sqrt{(1+g)^2 + \lambda^4 + 2\lambda^2 (1+g)}}{a\left[(1+g)^2 + \lambda^2\right]} = \frac{\left[(1+g) + \lambda^2\right]}{a\left[(1+g)^2 + \lambda^2\right]} \tag{44}$$

and at $\phi = \pi/2$ $\cos \lambda \theta = 0$; $\sin \lambda \theta = 1$

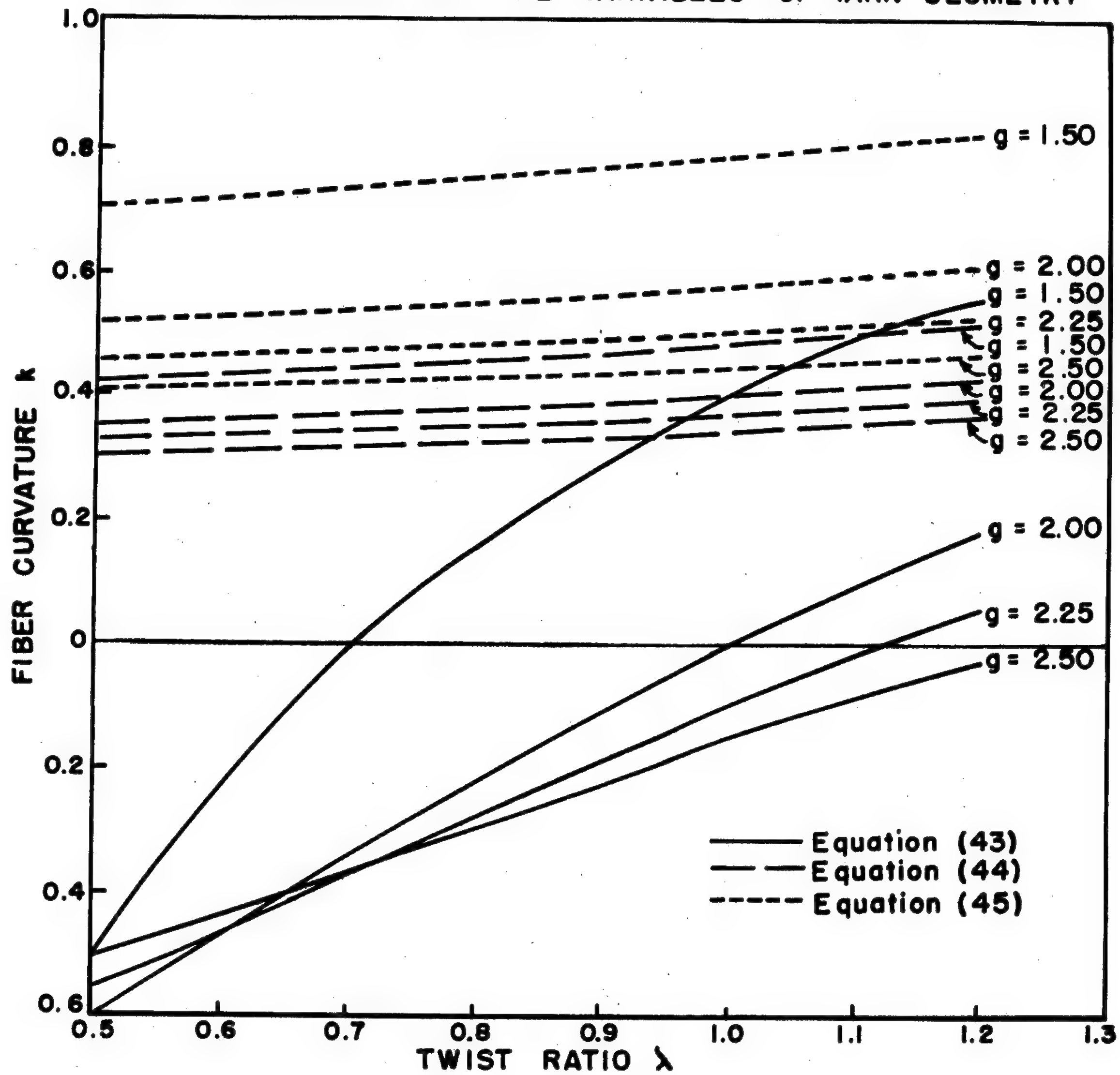
$$k = \frac{\sqrt{g^2 + \lambda^4 + 4\lambda^2 - \frac{g^2 \lambda^2}{g^2 + \lambda^2}}}{a\,(g^2 + \lambda^2)}$$

$$= \frac{\sqrt{\frac{g^4}{g^2 + \lambda^2} + \lambda^4 + 4\lambda^2}}{a\,(g^2 + \lambda^2)} \tag{45}$$

Finally in Figure 10 the dependance of k on λ for several values of g or r/a is illustrated for the points $\phi = 0$, $\pi/2$ and π. It should be emphasized that in dealing with cabled structures the reciprocal of k computed for the major element of the basic structure becomes the bending radius r in computations of strains set up in the sub-element. For example, in a plied structure whose single twist is hard but whose ply twist is soft, bending will impose little tensile strain on the singles since they are relatively free to move about. However, the change in curvature of the singles yarn axes, brought about by the bending, will impose tensile strains in individual fibers since they are restricted from moving freely within the singles yarn. In this latter calculation of yarn geometry r is taken as $1/k$ of the previous calculation.

FIG. 10

FIBER CURVATURE vs. THE VARIABLES OF YARN GEOMETRY



E. The Geometric Parameters in Textile Terms

Textile designers seldom work with data relating to twist ratios, or radius ratios. Fortunately, the dimensionless parameters used throughout this report can be readily computed from more practical yarn data. Equations (15) (16) and (17) illustrate the simplicity of the conversions. To facilitate computations, however, it was deemed advisable to reduce (17) to graphical form and designate the radius ratios in terms of yarn number ratios according to the relationship:

$$\frac{r}{a} = \sqrt{\frac{N_w}{N_f}} + 1 \qquad (46)$$

where the warp yarn of count N_w is considered bent around the filling yarn of count N_f. This has been done in Figures 11 & 12, which may be used in connection with Figures 3, 4, 5, and 10 to determine differences in path length, average strains, local helix angles, and fiber curvatures in terms of yarn counts and twist multiples.

It should be emphasized that this geometric analysis does not allow for flattening of one yarn as it is bent about the other. Flattening will alter the geometry of the yarn being bent, but the greatest effect, as Platt points out, will be due to an increase in the radius of curvature of the warp and filling yarns at their point of contact. It can be shown that this new radius (corresponding to $r - a$ in Figure 1) will be equal to the original radius of curvature divided by the cube of the flattening coefficient.

F. Textile Yarns and Wire Ropes

Civil engineers have studied the distribution of bending stresses in wire ropes used for heavy duty hauling. Since 1912 several analytical solutions of the geometric structure of twisted cables subjected to bending have been proposed by European engineers. One such solution based on the kinematic method was first used by Woernle in 1912 and was more recently amplified by Czitary [9]. It is of interest to compare the assumptions and results of Czitary's study with the analyses of this report.

FIG. 11

TWIST RATIO, λ, IN TERMS OF YARN NUMBER RATIOS & TWIST MULTIPLES.

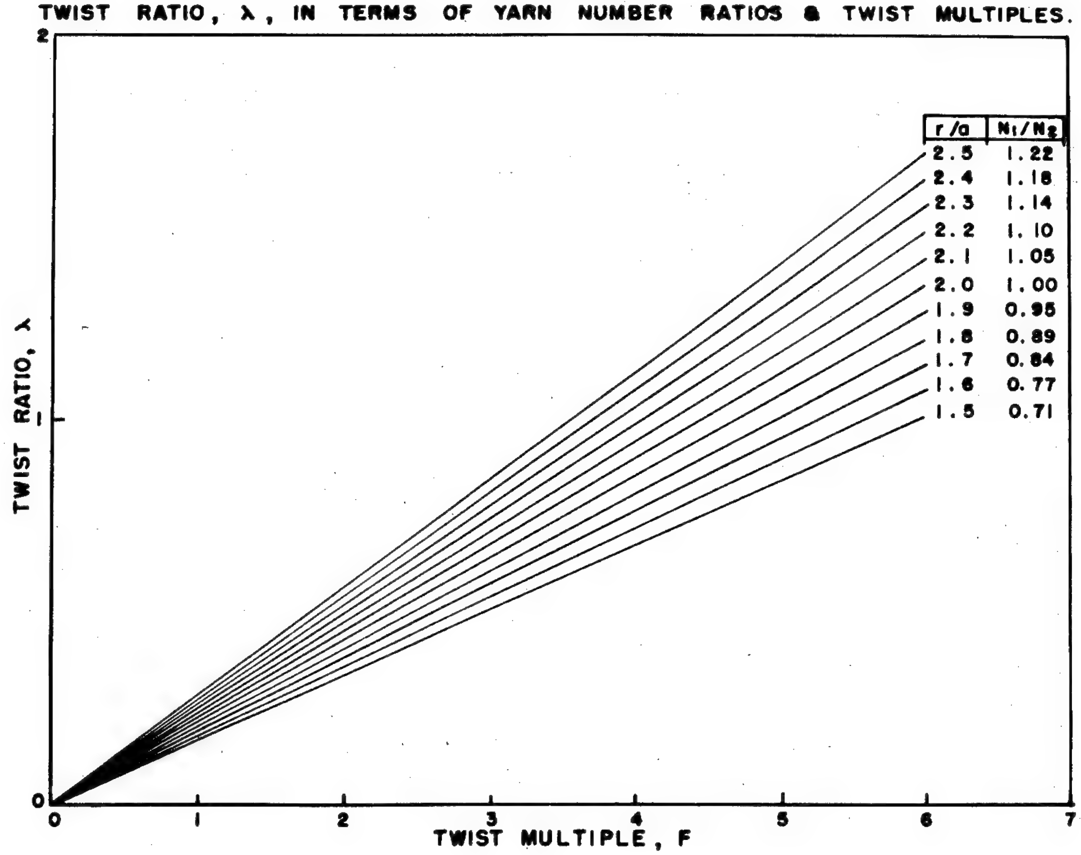
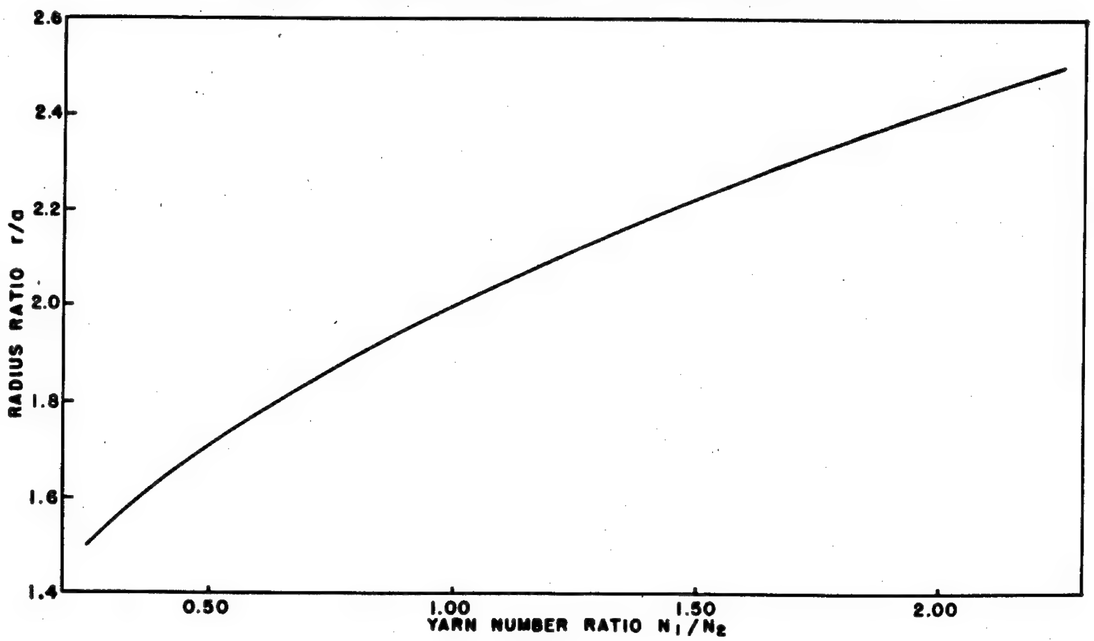


Fig. 12

RADIUS RATIOS IN TERMS OF YARN NUMBER RATIOS.



The kinematic method is based on assumptions similar to those of the above analysis. In kinematic terms the cable is considered to form a torus whose cross sections are circular and plane. The ratio of ϕ to θ is considered to be constant as in (3). The method pictures a particle moving along the wire strand as it lies in the bent cable. The motion of the particle consists of: 1) constant velocity about the periphery of a circle comprising the plane section of the torus (around circle HQP of Figure 1); and 2) constant angular velocity of this plane section about the center of curvature of the cable axis, i.e. the torus center.

Czitary derived an expression for the differential arc length along the particle path (strand axis) in terms of the radii of cable and torus, the angle θ and the twist ratio λ. Integrating and discarding terms of lesser magnitude, he obtains:

$$\int_{\phi=\frac{\pi}{2}}^{\phi=\pi} dS = \frac{r\pi}{2\lambda} + \frac{a}{\lambda} + \frac{\lambda a^2 \pi}{4r} \qquad (47)$$

representing the arc length of the outside quarter loop, and for the inside quarter loop:

$$\int_{\phi=0}^{\phi=\frac{\pi}{2}} dS = \frac{r\pi}{2\lambda} - \frac{a}{\lambda} + \frac{\lambda a^2 \pi}{4r} \qquad (48)$$

Subtracting (48) from (47) he shows the difference between outside and inside quarter loops to be

$$\frac{\Delta S}{2} = \frac{2a}{\lambda} \qquad (49)$$

as contrasted to (26). If however, $r \gg a$ and $aT \ll 1$, λ^2 will be small compared to $(r/a)^2$ and (26) will reduce to (49). In most textile uses, however, r is of the same order of magnitude as a and the product of yarn radius and twist is not small relative to unity. Equation (26) should therefore be used in preference to (49) in textile studies.

Adding (47) and (48) we have the half loop length in the bent yarn:

$$\int_{\phi=0}^{\phi=\pi} dS = \frac{r\pi}{\lambda} + \frac{\lambda a^2 \pi}{2r} \tag{50}$$

corresponding to (22) for the straight yarn. If (22) is expressed as:

$$\int_{\phi=0}^{\phi=\pi} dS = \frac{\pi r}{\lambda}\left[1 + \left(\frac{a\lambda}{r}\right)^2\right]^{1/2} \tag{51}$$

and use is made of the binomial theorem where

$$\sqrt{1+\chi} = 1 + \frac{\chi}{2} - \frac{1}{8}\chi^2 + \frac{1}{16}\chi^3 + \dots\dots$$

taking $\chi = \left(\frac{a\lambda}{r}\right)^2 \ll 1$, then the half loop length in the straight yarn becomes:

$$\int_{\phi = 0}^{\phi = \pi} dS = \frac{\pi r}{\lambda} + \frac{\lambda a^2 \pi}{2r} \qquad (52)$$

corresponding to (50) for the bent yarn. Czitary concluded that the increase in length of the outside of the helix turn is taken care of by the decrease in length of the inside helix turn. This agrees with the previously stated conclusion [1,2] that, within specified limits the length of the fiber loop in the bent yarn equals that of the fiber loop in the yarn before bending. Chow's [2] limitations on this equality are specific for he shows that the invariance of loop length depends on the condition that:

$$\frac{r}{a} - \sqrt{2 \left(\frac{r}{a}\right)^2 + \lambda^2} < -1 \qquad (53)$$

If λ is of negligible magnitude r/a must exceed 2.42 to satisfy (53), or conversely when

$\frac{r}{a} = 1.50$; must exceed 1.32

$\frac{r}{a} = 1.75$; must exceed 1.19

$\frac{r}{a} = 2.00$; must exceed 1.00

$\frac{r}{a} = 2.25$; must exceed .40

$\frac{r}{a} = 2.50$; must exceed .00

The limitations imposed by (53) apply as well to the integration of (18) to give (20). The concept of average strain introduced in (20) and (21) is likewise limited by (53). Nevertheless, it was considered advisable to plot the entire practical (in textile terms) range of values for $\Delta S/2$ and $\bar{\mathcal{E}}$ in Figures 3 and 4 for the purpose of achieving continuity and to permit estimation of orders of magnitude. If high accuracy is desired for values of average strain one may graphically integrate the local strain plotted against ϕ and divide by the total change in ϕ in Figure 2. Where the conditions of (53) are not met, the $\mathcal{E}$ versus ϕ curve between $\phi = 0$ and 45° will not equal the curve between $\phi = 45^{\circ}$ and 90° as is evident in Figure 2. As would be expected from (53) the curves of Figure 2 show increased symmetry as r/a is increased.

The kinematic method has been used to determine local helix angles. In effect the velocity of the particle at a given moment is the vector sum of its velocity components in the circle HQP of Figure 1 and around the torus due to movement of the plane section. The tangent of the angle between this velocity vector and the cable axis is simply the ratio of the two velocities, and can be shown to equal:

$$\tan \alpha = \tan Q \frac{r}{r - a \cos \phi} \qquad (54)$$

From (13)

$$\tan Q = \frac{a \lambda}{r}$$

whence agreement with (33) is reached. Carrying the kinematic approach further it has been shown that increased lateral strand space is not needed for low twist cables because of change in local helix angles during bending. This follows from the fact that changes in the major axes of the elliptical sections of the inclined strands in the outside loop are equal and opposite to the changes in the inner half of the loop. It follows that in low twist structures there is no lateral stress developed around the small circle perimeter. This point bears further consideration in the study of densely packed high twist structures.

To determine local curvatures Czitary determines 1) the normal acceleration of the particle due to rotation in the circle HQP (Fig. 1), 2) the normal acceleration due to rotation of the circle HQP about the axis O, and 3) the Coriolis acceleration parallel to the axis of the cable due to the angular velocity of the circular section HQP about the axis O and the component of relative velocity perpendicular to axis O. Czitary then determines the projections of these accelerations onto two mutually perpendicular planes both containing the tangent to the strand, or particle path at P. One plane is the oscillating plane of the strand path while the other contains the binormal of the path. In each plane the components of acceleration perpendicular to the path tangent at P are used to obtain the curvature of the path projection, according to the relationship:

$$\dot{v}_r = \frac{v^2}{\rho}$$

where $\dot{v}_r$ is the radial component of acceleration in each plane, ρ is the radius of curvature and v is the particle velocity. The curvatures in the two perpendicular planes are combined to furnish the local curvature of the strand. The change in curvature due to bending was determined to be:

$$\Delta k = \frac{\cos \alpha}{a\,(r - a \cos \phi)} \left[(a \cos \phi)^2 \cos^2 \alpha + a^2 (1 - \cos^2 \phi)(1 + \sin^2 \alpha)^2 \right]^{1/2} \tag{55}$$

Equation (55) is based on the relationship between curvature and bending moment components in a wire cross section, on the assumption that each component acts independently and on vectorial combination of the moments to provide the resultant curvature changes.

At $\phi = 0$ (55) reduces to

$$\Delta k = \frac{(g - 1)}{a \left[\lambda^2 + (g-1)^2 \right]} \tag{56}$$

and at $\phi = \pi$ to

$$\Delta k = \frac{(g+1)}{a\left[\lambda^2 + (g+1)^2\right]} \tag{57}$$

Now the curvature of the strand of the unbent cable or yarn is

$$k = \frac{\lambda^2}{a(g^2 + \lambda^2)} \tag{58}$$

which when subtracted from (43) and (44) provides for $\phi = 0, \pi$ respectively

$$\Delta k = \frac{\left[(1-g) + \lambda^2\right]}{a\left[(1-g)^2 + \lambda^2\right]} - \frac{\lambda^2}{a(g^2 + \lambda^2)} \tag{59}$$

$$\Delta k = \frac{\left[(1+g) + \lambda^2\right]}{a\left[(1-g)^2 + \lambda^2\right]} - \frac{\lambda^2}{a(g^2 + \lambda^2)} \tag{60}$$

If the ratio g or $r/a \gg 1$ as in the case of the bent cable the denominators of the second terms in (59), (60) equal those in the first terms and (59) and (60) reduce to (56) and (57). The change in curvature at $\phi = \pi/2$ is small and the subtraction of of (58) from (45) does not reduce so readily to (55) but rather to

$$\Delta k = \frac{\sqrt{(g^2 + 2\lambda^2)^2 + \lambda^4(g^2 + \lambda^2)}}{a\left[g^2 + \lambda^2\right]^{3/2}} - \frac{\sqrt{\lambda^4(g^2 + \lambda^2)}}{a\left[g^2 + \lambda^2\right]^{3/2}} \tag{61}$$

while (56) becomes

$$\Delta k = \frac{(g^2 + 2\ \lambda^2)}{a\ (g^2 + \ \lambda^2)^{3/2}} \qquad (62)$$

However, if $r \gg a$ and $aT \ll 1$, (61) will approach (62) showing the general agreement of the two expressions for the special case of extremely low twist structures bent into large radii of curvatures. For the average textile case (45) remains the more accurate expression for the determination of bending curvature in individual fibers.

The final subject of Czitary's paper has to do with the torsion of wire in bent cables. He assumes the wires to be free of torsion during the manufacturing process while the difference in torsional strain between straight and bent forms is accounted for in a manner similar to length changes in the upper and lower loops. In other words, the torsional strains required in one half of the loop is provided by the other half and there is no net twisting as a result. This is evidenced by observations of the rotation of a wire in a cable subjected to bending. To what extent this required rotation is hindered in densely packed highly twisted textile structures is not known, but it is safe to say that its effects are small compared to the tensile strains which occur due to restrictions on the longitudinal shifting of lengths from lower to upper loops during bending.

G. Summary and Conclusions

This report has outlined the results of a geometric analysis of the idealized structure of a bent yarn. The importance of such knowledge in problems dealing with the mechanical properties of twisted, woven and knitted textile structures has been stressed. Specific attention has been given to computations of local and average fiber strains which occur in highly twisted dense structures subjected to bending. The relative motion between fibers in both loose and tightly packed yarns has been considered, and finally the effect of bending on local fiber curvatures has been studied.

It is clear that the analysis has taken into account only the extreme cases of complete freedom or lack of freedom of motion between fibers during bending of the yarn. In practice the packing

density of the yarn and the frictional behavior of the fibers will determine the relative length transfers between upper and lower loops halves. The length transfers will influence the resultant local strains and therefore the fiber tensile stresses in the upper half loop. These stresses in turn develop lateral yarn pressure components which determine the frictional resistance to further length transfers.

For maximum bending fatigue resistance and wrinkle resistance, minimum friction between fibers and an open yarn structure appears desirable. In this way tensile strains resulting from bending of twisted structures may be kept to a minimum. If more elastic fibers are used in a fabric structure, closer packing of the yarn structure is possible without exceeding the yield level of the fiber in yarn bending. Clearly some degree of interfiber friction and yarn density is desirable if sufficient fiber strain and therefore permanent set is to be achieved when inserting a crease during the pressing operation. This is believed to be the reason why tightly twisted worsted suitings can be given such durable creases as contrasted to loosely twisted woolens. It has been said that the observed differences in crease acceptance of worsted and woolens are due to the relative radii of curvature reached at the fold. This factor is no doubt significant but the analysis presented here demonstrates that freedom of fiber movement can entirely eliminate strain over a wide range of curvatures. Considerably more emphasis must therefore be given to the packing and frictional behavior of fibers within the yarns.

Reasoning along the lines indicated above may be extended to studies of internal friction in yarn, cord, and rope structures. Where it can be shown experimentally that the cause of material failure is internal abrasion, the textile structure involved may be redesigned with the aid of the $\Delta S/2$ graphs developed here, and dislocations of its adjacent elements during bending may be drastically reduced. In like manner, local rubbing due to changes in local helix angles may be controlled.

Cords and ropes are multiple textile structures for which bending generally does not induce tensile strains in the major structural components because of the relative freedom of movement between these major components. However, freedom of movement between the sub-components within the major components is often restricted. In such cases computations of the curvatures of the major components due to total bending of the structure is facilitated by the graphs presented in this report. These curvatures can be used in subsequent computations of the local tensile strains in fibers or yarns of the subcomponents.

It is clear that much empirical work is necessary to demonstrate the applicability of these analyses. Such experimentation will throw light on the important factors of yarn flattening during bending and of yarn consolidation under tension. Without doubt there will be cases where deviations from the idealized geometry of this report will dominate the textile structure. It is our belief, however, that the cases for which the idealized geometry will be of use are numerous. For such cases this study provides much-needed quantitative relationships.

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